A New Riccati Equation Rational Expansion Method and Its Application

Yong Chen[a,b,c,] and Qi Wang[c,d]

[a] Nonlinear Science Center, Department of Mathematics, Ningbo University, Ningbo 315211, China
[b] Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, China
[c] M. M. Key Lab, Chinese Academy of Sciences, Beijing 100080, China
[d] Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China

Reprint requests to Dr. Y. C.; E-mail: chenyong@nbu.edu.cn

Z. Naturforsch. **60a,** 1 – 6 (2005); received October 13, 2004

A Riccati equation rational expansion (RERE) method is presented by a general ansatz, which is a direct and unified algebraic method for constructing multiple and more general travelling wave solutions of nonlinear partial differential equations. It can be implemented in a computer algebraic system. The proposed method is applied to consider the shallow long wave approximation equation and leads to a rich variety of exact solutions, including rational form solitary wave solutions, triangular periodic wave solutions and rational wave solutions.

Key words: Travelling Wave Solution; Riccati Equation Rational Expansion Method; Rational Form Solitary Wave Solutions.

1. Introduction

In recent years, nonlinear partial differential equations (NPDEs) were widely used to describe many important dynamic processes in physics, mechanics, chemistry, biology, etc. In particular, the wave phenomena observed in fluids, plasmas and elastic media are often modelled by solitary wave solutions. The exact solution of NPDEs, even nonlinear ordinary differential equations (NODEs), in closed form, has been a major concern for mathematicians and physicists. With the development of soliton theory there has been a great amount of activities aiming at finding methods for exact solutions of nonlinear differential equations, such as the Bäcklund transformation, Darboux transformation, Cole-Hopf transformation, various tanh methods, variable separation approach, Painlevé method, similarity reduction method [1 – 14]. Among these, the tanh function method provides a straightforward and effective algorithm to obtain such particular solutions of many nonlinear differential equations. Recently, much work has been concentrated on the various extensions and applications of the tanh function method [6 – 14]. The appeal and success of these methods lies in the fact that one circumvents integration to get explicit solutions. This is possible because soliton solutions are essentially of a localized nature. Writing soliton solutions of a nonlinear equation as the sum of polynomials in exponential solutions [6 – 7] or as polynomials of some solutions of the Riccati equation [8 – 14], the equation is transformed into a nonlinear system of algebraic equations. This system can be solved with the help of symbolic computation. Generally speaking, an ansatz and a subequation is considered as the more appropriate, the more general and more formal solution of NPDEs it serves to obtain.

The present work is motivated by the desire to present a new subequation method, named Riccati equation rational expansion (RERE) method, by proposing a more general ansatz. It is more general than the ansatz in the tanh function method [6 – 7], the extended tanh function method [8], the improved extended tanh function method [9 – 11], the projective Riccati equation method [12] and the general projective Riccati equation method [13 – 14], because it can be used to obtained more types and general formal solutions, which not only lead to results which could also be obtained by using the above mentioned methods [6 – 14], but also other types of solutions. For illustration, we apply the RERE method to solve and successfully construct new and more general solutions including rational formal solitary wave solutions, triangular periodic solutions and rational wave solutions for the equation of the shallow long wave approximation (SLA), i. e.,

$$u_t - uu_x - v_x + \frac{u_{xx}}{2} = 0, \; v_t - u_x v - uv_x - \frac{v_{xx}}{2} = 0, \quad (1.1)$$

which was found by Whitham [15] and Broer [16]. The symmetries and conservation laws of the system (1.1) have been discussed by Kuperschmidt [17]. By using the homogenous balance method, Zhang [18] obtained multiple soliton solutions of the system. Yan and Zhang [19] used the sine-cosine method to obtain three families of soliton solutions. Chen and Zheng [20] used the generalized extended tanh-function method to construct new explicit exact solutions of the SLA equations. Wang *et al.* [21] used a general projective Riccati equation method to obtain new exact travelling wave solutions for the SLA equations.

This paper is organized as follows: In Section 2 we summarize the RERE method. In Section 3 we apply the RERE method to the SLA equations and derive many rational formal solitary wave solutions, triangular periodic solutions and rational wave solutions. Finally conclusions will be presented.

2. Summary of the Riccati Equation Rational Expansion Method

In the following we outline the main steps of our method:

Step 1. Given a nonlinear partial differential equation system for some physical fields $u_i(x,y,t)$ in three variables x, y, t,

$$F_i(u_i,u_{it},u_{ix},u_{iy},u_{itt},u_{ixt},u_{iyt},u_{ixx},u_{iyy},u_{ixy},\cdots)=0, \tag{2.1}$$

we use the wave transformation

$$u_i(x,y,t)=U_i(\xi),\quad \xi=k(x+ly+\lambda t), \tag{2.2}$$

where k, l and λ are constants to be determined later, the nonlinear partial differential equation (2.1) is reduced to a nonlinear ordinary differential equation (NODE):

$$G_i(U_i,U_i',U_i'',\cdots)=0. \tag{2.3}$$

Step 2. We introduce a new ansatz in terms of a finite rational formal expansion:

$$U_i(\xi)=a_{i0}+\sum_{j=1}^{m_i}\frac{a_{ij}\phi^j(\xi)+b_{ij}\phi^{j-1}(\xi)\sqrt{R+\phi^2(\xi)}}{(\mu_1\phi(\xi)+\mu_2\sqrt{R+\phi^2(\xi)}+1)^j}. \tag{2.4}$$

The new variable $\phi=\phi(\xi)$ shall satisfy the NODE

$$\phi'-(R+\phi^2)=\frac{\mathrm{d}\phi}{\mathrm{d}\xi}-(R+\phi^2)=0. \tag{2.5}$$

Here R, a_{i0}, a_{ij} and b_{ij} ($i=1,2,\cdots;\ j=1,2,\cdots,m_i$) are constants to be determined later.

Step 3. We define the degree of $U_i(\xi)$ as $D[U_i(\xi)]=n_i$, which gives rise to the degrees of other expressions as

$$\begin{aligned}&D[U_i^{(\alpha)}]=n_i+\alpha,\\&D[U_i^{\beta}(U_j^{(\alpha)})^s]=n_i\beta+(\alpha+n_j)s.\end{aligned} \tag{2.6}$$

From this we can determine the value of m_i in (2.4). If n_i is a nonnegative integer, then we first make the transformation $U_i=\omega^{n_i}$.

Step 4. Substitute (2.4) into (2.3) along with (2.5) and then set all coefficients of $\phi^i(\xi)(\sqrt{R+\phi^2(\xi)})^j$ of the resulting system's numerator ($i=1,2,\cdots;\ j=0,1$) to zero to get an over-determined system of nonlinear algebraic equations with respect to k, μ_1, μ_2, a_{i0}, a_{ij} and b_{ij} ($i=1,2,\cdots;\ j=1,2,\cdots,m_i$).

Step 5. Solving this over-determined system of nonlinear algebraic equations by use of Maple, we end up with the explicit expressions for k, μ_1, μ_2, a_{i0}, a_{ij} and b_{ij} ($i=1,2,\cdots;\ j=1,2,\cdots,m_i$).

Step 6. It is well known that the general solutions of (2.5) are:

1) if $R<0$,

$$\begin{aligned}&\phi(\xi)=-\sqrt{-R}\tanh(\sqrt{-R}\xi),\\&\phi(\xi)=-\sqrt{-R}\coth(\sqrt{-R}\xi),\end{aligned} \tag{2.7}$$

2) if $R=0$,

$$\phi(\xi)=-\frac{1}{\xi}, \tag{2.8}$$

3) if $R>0$,

$$\begin{aligned}&\phi(\xi)=\sqrt{R}\tan(\sqrt{R}\xi),\\&\phi(\xi)=-\sqrt{R}\cot(\sqrt{R}\xi).\end{aligned} \tag{2.9}$$

Thus, according to (2.2), (2.4), (2.7)–(2.9) and the conclusions in *Step 5*, we can obtain the following rational formal travelling-wave solutions of (2.1):

1) if $R<0$,

$$
\begin{aligned}
u_i &= a_{i0}+\sum_{j=1}^{m_i}\frac{a_{ij}\left(-\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right)\right)^j \pm b_{ij}\left(-\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right)\right)^{j-1}\sqrt{-R}i\,\mathrm{sech}\left(\sqrt{-R}\xi\right)}{\left(1-\mu_1\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right)\pm\mu_2\sqrt{-R}i\,\mathrm{sech}\left(\sqrt{-R}\xi\right)\right)^j},\\
u_i &= a_{i0}+\sum_{j=1}^{m_i}\frac{a_{ij}\left(-\sqrt{-R}\coth\left(\sqrt{-R}\xi\right)\right)^j \pm b_{ij}\left(-\sqrt{-R}\coth\left(\sqrt{-R}\xi\right)\right)^{j-1}\sqrt{-R}\,\mathrm{csch}\left(\sqrt{-R}\xi\right)}{\left(1-\mu_1\sqrt{-R}\coth\left(\sqrt{-R}\xi\right)\pm\mu_2\sqrt{-R}\,\mathrm{csch}\left(\sqrt{-R}\xi\right)\right)^j},
\end{aligned}
\tag{2.10}
$$

2) if $R=0$,

$$
u_i = a_{i0}+\sum_{j=1}^{m_i}\frac{(-1)^j\left(a_{ij}\mp b_{ij}\right)}{\left(\xi-\mu_1\pm\mu_2\right)^j},
\tag{2.11}
$$

3) if $R>0$,

$$
\begin{aligned}
u_i &= a_{i0}+\sum_{j=1}^{m_i}\frac{a_{ij}\left(\sqrt{R}\tan\left(\sqrt{R}\xi\right)\right)^j \pm b_{ij}\left(\sqrt{R}\tan\left(\sqrt{R}\xi\right)\right)^{j-1}\sqrt{R}\sec\left(\sqrt{R}\xi\right)}{\left(1+\mu_1\sqrt{R}\tan\left(\sqrt{R}\xi\right)\pm\mu_2\sqrt{R}\sec\left(\sqrt{R}\xi\right)\right)^j},\\
u_i &= a_{i0}+\sum_{j=1}^{m_i}\frac{a_{ij}\left(-\sqrt{R}\cot\left(\sqrt{R}\xi\right)\right)^j \pm b_{ij}\left(-\sqrt{R}\cot\left(\sqrt{R}\xi\right)\right)^{j-1}\sqrt{R}\csc\left(\sqrt{R}\xi\right)}{\left(1-\mu_1\sqrt{R}\cot\left(\sqrt{R}\xi\right)\pm\mu_2\sqrt{R}\csc\left(\sqrt{R}\xi\right)\right)^j},
\end{aligned}
\tag{2.12}
$$

where $\xi=k(x+ly+\lambda t)$ and $i=\sqrt{-1}$.

Remark: The ansatz proposed here is more general than the ansatz in the tanh function method [6–7], the extended tanh function method [8], the improved extended tanh function method [9–11], the projective Riccati equations method [12] and the general projective Riccati equations method [13–14]. For some specific values of the parameters in (2.4), the above methods are recovered by the RERE method.

These values are as follows:

1) Setting $\mu_1=\mu_2=b_1=0$, we just recover the solutions obtained by the extended tanh function method [8].

2) Setting $\mu_1=\mu_2=0$, we just recover the solutions obtained by the improved extended tanh function method [9–11];

3) Setting $\mu_1=0$ and $\mu_2\neq 0$, we just recover the solutions obtained by the Projective Riccati method [12–14];

4) The other solutions obtained here are, to our knowledge, all new formally exact solutions of NPDEs.

3. The Shallow Long Wave Approximate Equation

Let us consider the shallow long wave approximation (SLA), i. e. (1.1).

By considering the wave transformations $u(x,t)=U(\xi)$, $v(x,t)=V(\xi)$ and $\xi=k(x+\lambda t)$, we transform (1.1) to the form

$$
\begin{aligned}
&\lambda U'-UU'-V'+\frac{k}{2}U''=0,\\
&\lambda V'-(UV)'-\frac{k}{2}V''=0.
\end{aligned}
\tag{3.1}
$$

According to the proposed method, we expand the solution of (3.1) in the form

$$
\begin{aligned}
U(\xi) &= a_0+\sum_{j=1}^{m_u}\frac{a_j\phi^j(\xi)+b_j\phi^{j-1}(\xi)\sqrt{R+\phi^2(\xi)}}{\left(\mu_1\phi(\xi)+\mu_2\sqrt{R+\phi^2(\xi)}+1\right)^j},\\
V(\xi) &= A_0+\sum_{j=1}^{m_v}\frac{A_j\phi^j(\xi)+B_j\phi^{j-1}(\xi)\sqrt{R+\phi^2(\xi)}}{\left(\mu_1\phi(\xi)+\mu_2\sqrt{R+\phi^2(\xi)}+1\right)^j},
\end{aligned}
\tag{3.2}
$$

where $\phi(\xi)$ satisfies (2.5). Balancing the term V'' with term $(UV)'$ and the term V' with term UU' in (3.1) gives $m_u=1$ and $m_v=2$. So we have

$$
\begin{aligned}
U(\xi) &= a_0+\frac{a_1\phi(\xi)+b_1\sqrt{R+\phi^2(\xi)}}{\mu_1\phi(\xi)+\mu_2\sqrt{R+\phi^2(\xi)}+1},\\
V(\xi) &= A_0+\frac{A_1\phi(\xi)+B_1\sqrt{R+\phi^2(\xi)}}{\mu_1\phi(\xi)+\mu_2\sqrt{R+\phi^2(\xi)}+1}\\
&\quad+\frac{A_2\phi^2(\xi)+B_2\phi(\xi)\sqrt{R+\phi^2(\xi)}}{\left(\mu_1\phi(\xi)+\mu_2\sqrt{R+\phi^2(\xi)}+1\right)^2}.
\end{aligned}
\tag{3.3}
$$

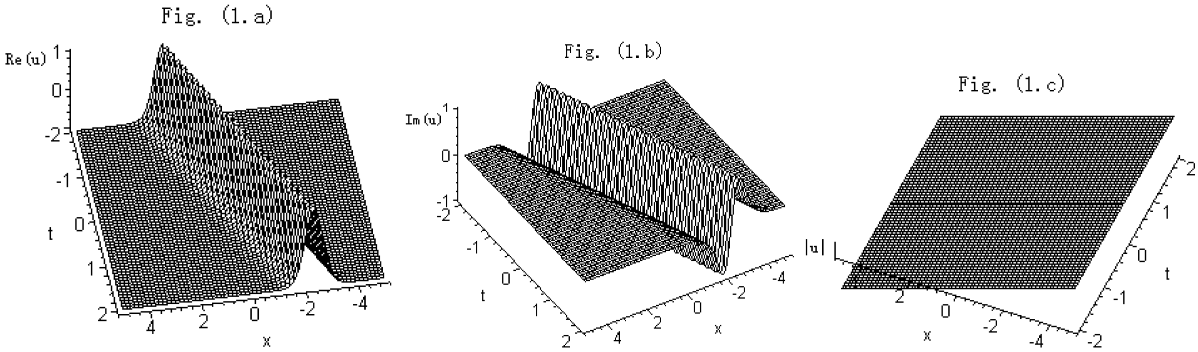

Fig. 1. The solitary wave solution u_1 of an SLA equation. Fig. 1a, the real part; Fig. 1b, the imaginary part and Fig. 1c, the modulus, where $R=-2$, $\lambda=1$, $b_1=2$.

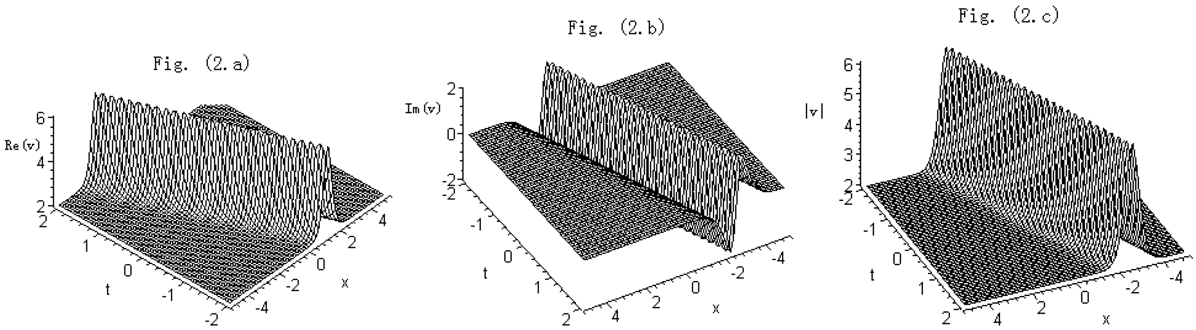

Fig. 2. The solitary wave solution v_1 of an SLA equation. Fig. 2a, the real part; Fig. 2b, the imaginary part and Fig. 2c, the modulus, where $R=-2$, $\lambda=1$, $b_1=2$.

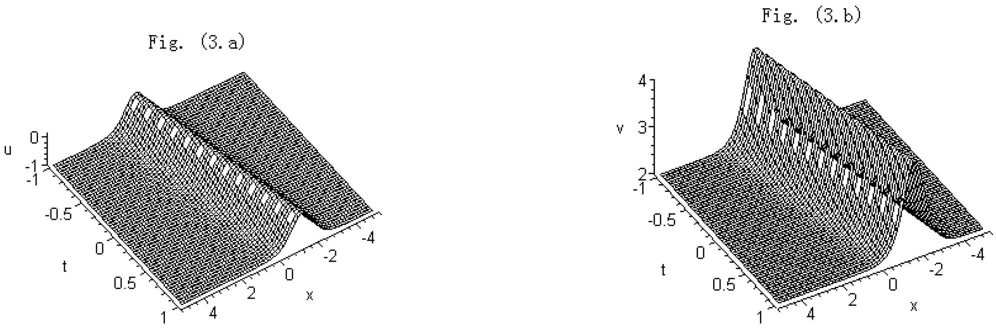

Fig. 3. The solitary wave solutions u_2 and v_2 of an SLA equation, where $R=-2$, $\lambda=1$, $b_1=2$.

Substituting (3.3) along with (2.5) into (3.1), yields a set of algebraic equations for $\phi^i(\xi)(\sqrt{R+\phi^2(\xi)})^j$, $(i=0,1,\cdots;\ j=0,1)$. Setting the coefficients of these terms $\phi^i(\xi)(\sqrt{R+\phi^2(\xi)})^j$ to zero yields a set of over-determined algebraic equations with respect to a_0, a_1, b_1, A_0, A_1, B_1 A_2, B_2, μ_1, μ_2 and k.

By use of the Maple software package "Charsets" by Dongming Wang, which is based on the Wu-elimination method [22], solving the over-determined algebraic equations, we get rich results. However, for simplification, we only choose one example obtained by solving the over-determined algebraic equations as illustration, and from that case we obtain a family of solutions as a characteristic example. At the same time, the properties of general solitary wave solutions are shown by some figures (Figs. 1–3).

Example case:

$$k=-2\frac{a_1}{R+1},\quad \mu_1=\pm1,\quad a_0=-\frac{a_1R-\lambda-\lambda R}{R+1},$$

$$\mu_2 = \pm 1, \quad b_1 = a_1, \quad A_0 = 2\frac{{a_1}^2 R}{(R+1)(R-1)}, \quad A_1 = B_1 = -4\frac{{a_1}^2 R}{(R+1)(R-1)}, \quad A_2 = B_2 = 2\frac{{a_1}^2}{R-1}. \tag{3.4}$$

From that case, we obtain the following family of solutions for (1.1):

$$\begin{aligned} u_1 &= \frac{-a_1 R + \lambda + \lambda R}{R+1} + \frac{-a_1\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right) \pm a_1\sqrt{-R}i\mathrm{sech}\left(\sqrt{-R}\xi\right)}{\pm\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right) \pm \sqrt{-R}i\mathrm{sech}\left(\sqrt{-R}\xi\right) + 1}, \\ v_1 &= 2\frac{{a_1}^2 R}{(R+1)(R-1)} + 4\frac{{a_1}^2 R\left(\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right) \pm \sqrt{-R}i\mathrm{sech}\left(\sqrt{-R}\xi\right)\right)}{(R^2-1)\left(\pm\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right) \pm \sqrt{-R}i\mathrm{sech}\left(\sqrt{-R}\xi\right) + 1\right)} \\ &\quad - 2\frac{{a_1}^2 R\left(\tanh^2\left(\sqrt{-R}\xi\right) \pm i\tanh\left(\sqrt{-R}\xi\right)\mathrm{sech}\left(\sqrt{-R}\xi\right)\right)}{(R-1)\left(\pm\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right) \pm \sqrt{-R}i\mathrm{sech}\left(\sqrt{-R}\xi\right) + 1\right)^2}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} u_2 &= \frac{-a_1 R + \lambda + \lambda R}{R+1} - \frac{a_1\sqrt{-R}\coth\left(\sqrt{-R}\xi\right) \pm a_1\sqrt{-R}\mathrm{csch}\left(\sqrt{-R}\xi\right)}{\pm\sqrt{-R}\coth\left(\sqrt{-R}\xi\right) \pm \sqrt{-R}\mathrm{csch}\left(\sqrt{-R}\xi\right) + 1}, \\ v_2 &= 2\frac{{a_1}^2 R}{(R+1)(R-1)} + 4\frac{{a_1}^2 R\left(\sqrt{-R}\coth\left(\sqrt{-R}\xi\right) \pm \sqrt{-R}\mathrm{csch}\left(\sqrt{-R}\xi\right)\right)}{(R^2-1)\left(\pm\sqrt{-R}\coth\left(\sqrt{-R}\xi\right) \pm \sqrt{-R}\mathrm{csch}\left(\sqrt{-R}\xi\right) + 1\right)} \\ &\quad - 2\frac{a_1^2 R\left(\coth^2\left(\sqrt{-R}\xi\right) \pm \coth\left(\sqrt{-R}\xi\right)\mathrm{csch}\left(\sqrt{-R}\xi\right)\right)}{(R-1)\left(\pm\sqrt{-R}\coth\left(\sqrt{-R}\xi\right) \pm \sqrt{-R}\mathrm{csch}\left(\sqrt{-R}\xi\right) + 1\right)^2}, \end{aligned} \tag{3.6}$$

where $\xi = k(x + \lambda t)$, k is determined by (3.4), $R < 0$, a_1 and λ are arbitrary constants.

4. Summary and Conclusions

In this paper we have presented the RERE method. The method leads to more solutions than the methods described in [6–14]. The SLA equation (1) is chosen to illustrate the method such that many new families such as solitary wave solutions, triangular periodic wave solutions and rational wave solutions are obtained. The algorithm can also be applied to other NPDEs. As mentioned in Section 2, further work about various extensions and improvements of the RERE method is necessary to find more general ansatzes or more general subequations. As we have seen, one needs to solve many algebraic equations to get the desired polynomial or rational solitary wave solutions (even for simple NPDEs or NODEs). This is the reason for the success of the symbolic mathematical computation discipline and of the computer software symbolic systems like Maple or Mathematica, which allow to perform complicated and tedious algebraic calculations.

Acknowledgements

The work was supported by the Zhejiang Provincial Natural Science Foundation of China under Grant No. Y604056, the Postdoctoral Science Foundation of China, and the National Outstanding Youth Foundation of China (No. 19925522).

[1] M.J. Ablowitz and P.A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, New York 1991.
[2] V.B. Matveev and M.A. Salle, Darboux Transformation and Solitons, Springer, Berlin 1991.
[3] Y. Chen, Z.Y. Yan, and H.Q. Zhang, Theor. Math. Phys. **132**, 970 (2002). Y. Chen, B. Li, and H.Q. Zhang, Commun. Theor. Phys. **39**, 135 (2003). Y. Chen, B. Li, and H.Q. Zhang, Chaos, Solitons and Fractals **17**, 693 (2003). B. Li, Y. Chen, and H.Q. Zhang, Phys. Lett. A **305**, 377 (2002). B. Li, Y. Chen, and H.Q. Zhang, Z. Naturforsch. **58a**, 464 (2003).
[4] S.Y. Lou and J.Z. Lu, J. Phys. A **29**, 4029 (1996). S.Y. Lou, Phys. Lett. A **277**, 94 (2000).
[5] R. Hirta and J. Satsuma, Phys. Lett. A **85**, 407 (1981).
[6] E.J. Parkes and B.R. Duffy, Comput. Phys. Commun. **98**, 288 (1996). E.J. Parkes and B.R. Duffy, Phys. Lett. A **229**, 217 (1997).
[7] W. Hereman, Comput. Phys. Commun. **65**, 143 (1991).
[8] E. Fan, Phys. Lett. A **277**, 212 (2000). E. Fan, Comput. Math. Appl. **43**, 671 (2002).

[9] Z. Y. Yan, Comput. Phys. Commun. **153**, 145 (2003). Z. Y. Yan and H. Q. Zhang , Phys. Lett. A **285**, 355 (2001).

[10] Y. Chen, B. Li, and H. Q. Zhang, Int. J. Mod. Phys. C **13**, 99 (2003). Y. Chen, Z. Y. Yan, and H. Q. Zhang, Phys. Lett. A **307**, 107 (2003).

[11] B. Li, Y. Chen, and H. Q. Zhang, J. Phys. A: Math. Gen. **35**, 8253 (2002). B. Li, Y. Chen, and H. Q. Zhang, Z. Naturforsch. **57a**, 874 (2002). B. Li, Y. Chen, and H. Q. Zhang, Appl. Math. Comput. **146**, 653 (2003).

[12] R. Conet and M. Musette, J. Phys. A: Math. Gen. **25**, 5609 (1992).

[13] Z. Y. Yan, Chaos, Solitons and Fractals **16**, 759 (2003).

[14] Y. Chen and B. Li, Chaos, Solitons and Fractals **19**, 977 (2004). Q. Wang, Y. Chen, and H. Q. Zhang, Commun. Theor. Phys. (Beijing, China) **41L**, 821 (2004).

[15] G. B. Whitham, Proc. R. Soc. London Sect. A **299**, 6 (1967).

[16] L. T. F. Broer, Appl. Sci. Res. **31**, 377 (1957).

[17] B. A. Kuperschmidt, Commun. Math. Phys. **99**, 51 (1985).

[18] J. F. Zhang, Acta Phys. Sin. **47**, 1416 (1998) (in Chinese).

[19] Z. Y. Yan and H. Q. Zhang, Acta Phys. Sin. **48**, 1962 (1999) (in Chinese).

[20] Y. Chen and Y. Zheng, Int. J. Mod. Phys. C **14**, 1 (2003).

[21] Q. Wang, Y. Chen, B. Li, and H. Q. Zhang, Appl. Math. Comput. **160**, 77 (2005).

[22] W. Wu, Algorithms and Computation, Springer, Berlin 1994, p. 1.